READING MATERIAL AND ASSINGNMENT

The Transition Matrix for change of coordinates

We have already discussed: An ordered basis for a vector space and a method for finding the coordinates of a vector with respect to a finite ordered basis (Known as coordinatization).

Now, How the coordinates of a vector change when we convert from one ordered basis to another.

For this, Transition Matrix can be used to change the coordinatization of a vector from one ordered basis to another ordered basis.

Definition Let V be a non-trivial n-dimensional vector space with ordered bases

$$\beta = (\beta_1, \beta_2, \ldots, \beta_n) \text{ \& } \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n).$$

Let P be the $n \times n$ matrix whose ith column, for $1 \leq i \leq n$ equal $[\beta_i]_\gamma$ where β_i is the ith basis vector in β. Then P is called the transition matrix from β- coordinates to γ-coordinates.

[Notation $P_{\gamma \leftarrow \beta}$]

$$\text{Matrix } P_{\gamma \leftarrow \beta} = \left[[\beta_1]_\gamma, [\beta_2]_\gamma \cdots [\beta_i]_\gamma \cdots [\beta_n]_\gamma \right]$$

Ex 6
P286
Hecker

Consider ordered bases

$\beta = ([-4, 5, -1, 0, -1], [1, -3, 2, 2, 5], [1, -2, 1, 1, 3])$

&

$\gamma = ([1, 0, -1, 0, 4], [0, 1, -1, 0, 3], [0, 0, 0, 1, 5])$

Such that they span the same subspace of V of $\mathbb{R}^5$. Find the transition matrix from γ to β.

Soln Consider the augmented matrix

$$\begin{array}{cccccc} \beta_1 & \beta_2 & \beta_3 & \gamma_1 & \gamma_2 & \gamma_3 \end{array}$$
$$\left[\begin{array}{ccc|c|c|c} -4 & 1 & 1 & 1 & 0 & 0 \\ 5 & -3 & -2 & 0 & 1 & 0 \\ -1 & 2 & 1 & -1 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ -1 & 5 & 3 & 4 & 3 & 5 \end{array}\right]$$

After row reducing, the reduced row echelon form

$$\sim \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -5 & -4 & -3 \\ 0 & 0 & 1 & 10 & 8 & 7 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right]$$

This gives $[\gamma_1]_\beta = \begin{bmatrix} 1 \\ -5 \\ 10 \end{bmatrix}$, $[\gamma_2]_\beta = \begin{bmatrix} 1 \\ -4 \\ 8 \end{bmatrix}$, $[\gamma_3]_\beta = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$

Then vectors form the columns of the transition matrix from γ to β.

i.e. $P_{\beta \leftarrow \gamma} = \begin{bmatrix} 1 & 1 & 1 \\ -5 & -4 & -3 \\ 10 & 8 & 7 \end{bmatrix}$

Method for Calculating a Tansition Matrix

(Transition Matrix Method)

Suppose $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ & $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ are ordered bases for a non-trivial k-dimensional subspace of $\mathbb{R}^n$.

To find the transition matrix P from α to β (i.e. $P_{\beta \leftarrow \alpha}$), reduce the following Augmented matrix to its reduced row-echelon form

$$\begin{array}{c} \begin{array}{cccc|cccc} \text{1st column} & \text{2nd column} & \cdots & \text{kth column} & \text{1st vector in } \alpha & \text{2nd vector in } \alpha & & \text{kth vector in } \alpha \end{array} \\ \left[\begin{array}{cccc|cccc} \beta_1 & \beta_2 & & \beta_k & \alpha_1 & \alpha_2 & \cdots & \alpha_k \end{array}\right] \end{array}$$

Reduced row echelon form

$$\sim \left[\begin{array}{c|c} I_k & P \\ \text{rows of} & \text{zeros} \end{array}\right] \qquad P = P_{\beta \leftarrow \alpha}$$

Ex 7 P287 Hacker

Consider
$$B = \left(\begin{bmatrix} 7 & 3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right)$$
and
$$C = \left(\begin{bmatrix} 22 & 7 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 12 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 33 & 12 \\ 0 & 2 \end{bmatrix} \right)$$
be the ordered bases of a subspace of $M_{2\times 2}$.
Find the transition matrix from B to C.
(i.e. $P_{C \leftarrow B}$)

Soln First convert the matrices in B and C into vectors in $\mathbb{R}^4$:

$$B = ([7, 3, 0, 0], [1, 2, 0, -1], [1, -1, 0, 1])$$
$$C = ([22, 7, 0, 2], [12, 4, 0, 1], [33, 12, 0, 2])$$

Construct the matrix

$$\begin{array}{cccccc} C_1 & C_2 & C_3 & B_1 & B_2 & B_2 \end{array}$$
$$\left[\begin{array}{ccc|ccc} 22 & 12 & 33 & 7 & 1 & 1 \\ 7 & 4 & 12 & 3 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & -1 & 1 \end{array}\right]$$

After reduction, the reduced row echelon form

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & -4 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \therefore P_{C\leftarrow B} = \begin{bmatrix} 1 & -2 & 1 \\ -4 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Ex Let $\beta = \left(\begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -4 & 1 \end{bmatrix} \right)$

and $\gamma = \left(\begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & -4 \\ -7 & 4 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \right)$

be the ordered bases for a subspace of $M_{m \times n}$.

Find the transition matrix from β to γ. (i.e. $P_{\gamma \leftarrow \beta}$).

Soln Convert the matrices in β & γ into vectors in $\mathbb{R}^4$.

Then

$$\beta = \left([1, 3, 5, 1], [2, 1, 0, 4], [3, 1, 1, 0], [0, 2, -4, 1] \right)$$

& $$\gamma = \left([-1, 1, 3, -1], [1, 0, 0, 1], [3, -4, -7, 4], [1, -1, -2, 1] \right)$$

Construct the matrix

$$\left[\begin{array}{cccc|cccc} -1 & 1 & 3 & 1 & 1 & 2 & 3 & 0 \\ 1 & 0 & -4 & -1 & 3 & 1 & 1 & 2 \\ 3 & 0 & -7 & -2 & 5 & 0 & 1 & -4 \\ -1 & 1 & 4 & 1 & 1 & 4 & 0 & 1 \end{array}\right]$$

reduced row echelon form

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & -4 & 2 & -9 \\ 0 & 1 & 0 & 0 & 4 & 5 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 & 2 & -3 & 1 \\ 0 & 0 & 0 & 1 & -4 & -13 & 13 & -15 \end{array}\right]$$

$$\therefore \quad P_{\gamma \leftarrow \beta} = \begin{bmatrix} -1 & -4 & 2 & -9 \\ 4 & 5 & 1 & 3 \\ 0 & 2 & -3 & 1 \\ -4 & -13 & 13 & -15 \end{bmatrix}$$

The next theorem shows that the transition matrix can be used to change the coordinatization of a vector v from one ordered basis β to another ordered basis γ, i.e. if we have

$[v]_\beta$ then $[v]_\gamma$ can be obtained

by using the transition matrix from β to γ

$(P_{\gamma \leftarrow \beta})$

Theorem 4.20
P 288
Hacker

Let β and γ be the ordered bases for a non trivial n-dimensional vector space V, and let P be $n \times n$ transition matrix from β to γ iff for every $v \in V$,

$$P[v]_\beta = [v]_\gamma$$

Proof: Read from the book.

Ex 8
P288

Let $\beta = \left(\begin{bmatrix} 7 & 3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right)$ and

$\gamma = \left(\begin{bmatrix} 22 & 7 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 12 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 33 & 12 \\ 0 & 2 \end{bmatrix} \right)$

be the ordered bases for a subspace V of $M_{2\times 2}$.

Find (1) $P_{\gamma \leftarrow \beta}$.

(2) Find $P[v]_\beta$ where $v = \begin{bmatrix} 25 & 24 \\ 0 & -9 \end{bmatrix} \in V$.

Soln We have already found that the transition matrix P from β to γ (i.e. $P_{\gamma \leftarrow \beta}$)*

*(see Ex 7)

$$P_{\gamma \leftarrow \beta} = \begin{bmatrix} 1 & -2 & 1 \\ -4 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Now $v = \begin{bmatrix} 25 & 24 \\ 0 & -9 \end{bmatrix}$

$= [25, 24, 0, -9]$ (converting into vector in $\mathbb{R}^4$)

For $[v]_\beta$, construct the matrix:

$$\left[\begin{array}{ccc|c} 7 & 1 & 1 & 25 \\ 3 & 2 & -1 & 24 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -9 \end{array}\right]$$

Reduce it to its reduced row echelon form:

$R_1 \to R_1 - 2R_2$, $R_3 \leftrightarrow R_4$ $\quad\quad$ $R_2 \to R_2 - 3R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 3 & -23 \\ 3 & 2 & -1 & 24 \\ 0 & -1 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 3 & -23 \\ 0 & 11 & -10 & 93 \\ 0 & -1 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$R_1 \to R_1 - 3R_3$ $\quad\quad$ $R_2 \to R_2 + 10R_3$ $\quad\quad$ $R_3 \to (-1)R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 11 & -10 & 93 \\ 0 & -1 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & -1 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & -1 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$ $\quad\quad$ $R_3 \to (-1)R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -6 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore [v]_\beta = \begin{bmatrix} 4 \\ 3 \\ -6 \end{bmatrix}$$

Now $P[v]_\beta = \begin{bmatrix} 1 & -2 & 1 \\ -4 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} -8 \\ -19 \\ 13 \end{bmatrix}$

Note: $\underset{\gamma\leftarrow\beta}{P}[v]_\beta = [v]_\gamma$ (by thm 4.20)

One can easily verify that

$$\begin{bmatrix} 25 & 24 \\ 0 & -9 \end{bmatrix} = 4\begin{bmatrix} 7 & 3 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} - 6\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= -8\begin{bmatrix} 22 & 7 \\ 0 & 2 \end{bmatrix} - 19\begin{bmatrix} 12 & 4 \\ 0 & 1 \end{bmatrix} + 13\begin{bmatrix} 33 & 12 \\ 0 & 2 \end{bmatrix}$$

The next theorem states that the joint effect of two transitions between bases can be represented by the product of the transition matrices in the reverse order.

Theorem 4.21 P289 Suppose B, C and D are ordered bases for a non-trivial finite dimensional vector space V. Let P be the transition matrix from B to C, & let Q be the transition matrix from C to D. Then QP is the transition matrix from B to D.

Ex suppose B, C & D are ordered bases for some subspace V of $\mathbb{R}^3$ given by

$B = ([1, 2, 2], [3, 7, 8], [3, 9, 13])$,

$C = ([1, 4, 1], [2, 1, 0], [1, 0, 0])$

and $D = ([7, -3, 2], [1, 7, -3], [1, -2, 1])$

let transition matrix

$$P = P_{C \leftarrow B}$$

& the transition matrix

$$Q = Q_{D \leftarrow C}$$

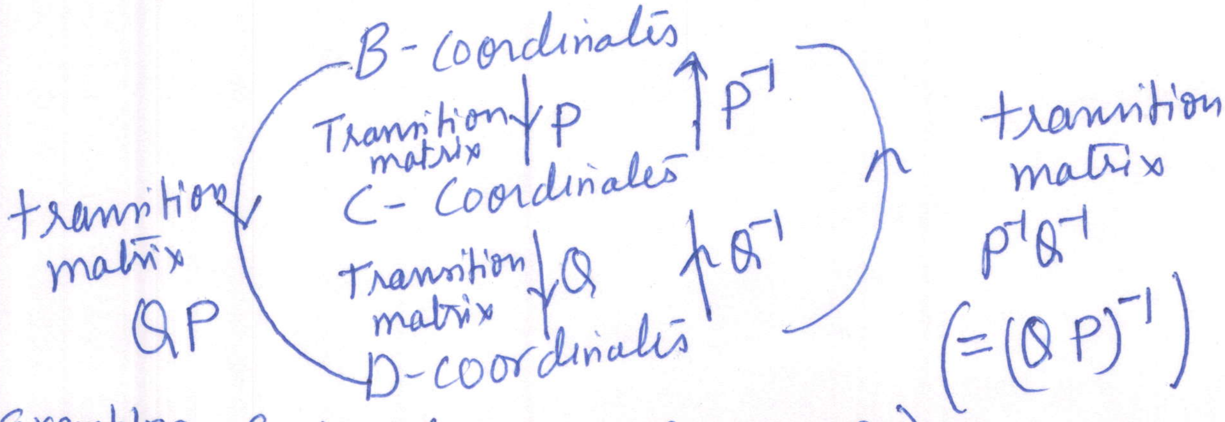
then $T = QP$

is the transition matrix $T_{D \leftarrow B}$

Find P, Q & T and
Verify the above.

Next theorem shows how to reverse a transition from one basis to another.

<u>Theorem 4.22</u>
<u>P 290 Hacker</u> Let B and C be ordered bases for a non-trivial finite dimensional vector space V, and let P be the transition matrix from B to C $(P_{C \leftarrow B})$. Then P is non-singular, and P^{-1} is the transition matrix from C to B $(P^{-1}_{B \leftarrow C})$.



→ Do Examples 9, 10 (pages 289, 290-91) Hacker.

Diagonalization and the Transition Matrix

When the diagonalization method is performed on a $n \times n$ matrix A, the matrix P obtained in this process turns out to be a transition matrix from B-coordinates to standard coordinates, where B is an ordered basis for $\mathbb{R}^n$ consisting of fundamental eigen vectors for A.

Ex 11
P291 Hacker

Consider $A = \begin{bmatrix} 14 & -15 & -30 \\ 6 & -7 & -12 \\ 3 & -3 & -7 \end{bmatrix}$

Characteristic poly. eqn is $p_A(\lambda) = 0$

i.e. $|\lambda I - A| = 0$

$\lambda^3 - 3\lambda - 2 = 0$

$\lambda = 2, -1, -1$

For $\lambda = 2$, fundamental ch. vector $v_1 = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$

For $\lambda = -1$ fundamental ch. vectors

$v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ & $v_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Now

$B = (v_1\ v_2\ v_3)$ forms a basis for $\mathbb{R}^3$.

(As $\{v_1, v_2, v_3\}$ is a l. indep set & $\dim \mathbb{R}^3 = 3$)

Let $S = (\hat{i}, \hat{j}, \hat{k})$ be the standard basis for $\mathbb{R}^3$.

$P_{S \leftarrow B}$ – transition matrix from B to S

then $$P_{S \leftarrow B} = \begin{bmatrix} 5 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad \begin{array}{ccc|ccc} \hat{i} & \hat{j} & \hat{k} & & & \\ \end{array}$$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 1 & 2 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

Now
$$P^{-1} = \begin{bmatrix} 1 & -1 & -2 \\ -2 & 3 & 4 \\ -1 & 1 & 3 \end{bmatrix}.$$

then $$P_{B \leftarrow S} = P^{-1}$$

Recall $$P^{-1} A P = \text{Diagonal matrix } D$$
$$= \begin{bmatrix} 2 & 0 & 0 \\ 6 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

———— x ————

Q.1 Let $S = (V_1, V_2, V_3, V_4)$ be an ordered basis for $\mathbb{R}^4$, where

$V_1 = [1, 1, 0, 0]$, $V_2 = [2, 0, 1, 0]$,

$V_3 = [0, 1, 2, -1]$, $V_4 = [0, 1, -1, 0]$.

If $V = [1, 2, -6, 2]$ compute $[V]_S$

Q2 Let $S = ([2,0,1], [1,2,0], [1,1,1])$

and $T = ([6,3,3], [4,-1,3], [5,5,2])$

be the ordered bases for $\mathbb{R}^3$.

(i) compute the transition matrix $P_{S \leftarrow T}$

(i.e. from T-basis to S-basis)

(ii) Show that $[V]_S = P_{S \leftarrow T} [V]_T$

where $V = [4, -9, 5]$

Q3 Let $S = ([1, 2], [0, 1])$ and

$T = ([1, 1], [2, 3])$

be the ordered bases for $\mathbb{R}^2$.

Let $V = [1, 5]$ & $W = [5, 4]$, find

(i) $[V]_T$, $[W]_T$

(ii) $P_{S \leftarrow T}$

(iii) $[V]_S$, $[W]_S$ using $P_{S \leftarrow T}$

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Q4 Let $S = ([-1,2,1], [0,1,1], [-2,2,1])$

& $T = ([-1,1,0], [0,1,0], [0,1,1])$

be the ordered bases for $\mathbb{R}^3$. and

$$[v]_S = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Determine $[v]_T$ using transition matrix

$P_{T \leftarrow S}$.