Mathematics of Finance

ASHA RANI
B.Com. (Hons.) IV Semester
Section H
Business Mathematics

Session Details

This Session Covers

- Simple and compound Interest rate
- Effective and Normal Rate of Interest
- Present value of Amount
- Equation of Value
- Depreciation

Solved Illustrations and Practice Handouts are provided separately

Simple Interest

Simple interest is the interest computed on the principal amount borrowed . If I denotes the interest rate and P denote the principal (in Rupees term) at an interest rate of r per year for t years. Then, we can say

- Interest = P * r * t

The accumulated amount A, is the sum of principal and Interest after t years can be written as

- A= P + Prt = P(1 +rt)

Compound Interest

Compounding is the repetitive process of earning (or paying) interest, adding that interest to the principal balance, and earning even more interest in the next round due to that increased account balance i.e principal plus interest of earlier period.

Frequency of Compounding may not be annually, need to use the rate per compounding period as i, and calculate the number of compounding periods as n.

It is also possible in some cases that rate of compounding and nature of compounding keep changing from one period to other. In that case we have to compute the amount accordingly considering the changed rate and frequency.

Compound Interest Formula

$$A = P\left(1+i\right)^n$$

A = final amount including principal

P = principal amount

i = interest rate per year

n = number of years invested

Asha Rani

Difference between

SIMPLE INTEREST & COMPOUND INTEREST

Interest

Money ←

BANK

Borrower — Lender

Money + Interest % →

Simple Interest

$SI = P \times R \times T$

$P = 1,00,000$
$R = 9\%$
$T = 2$ years

$SI = 1,00,000 \times \frac{9}{100} \times 2 \Rightarrow 18,000$

Total amount to be paid = P + SI => 1,00,000 + 18,000

$\underline{1,18,000}$

Compound Interest

→ 2 or more elements

Principal & Rate of interests

$P = 1,00,000$
$R = 9\%$
$T = 2$ years

$CI = P \times (1+R)^n$

$CI = 1,00,000 \times \left(1 + \frac{9}{100}\right)^2$

$CI = 1,00,000 \times (1.09)^2$

$CI = 1,00,000 \times 1.09 \times 1.09$

$CI = \underline{1,18,810}$

Derivation of Continuously Compounding formula

Amount formula in case of normal compounding, $A = P\left(1+\frac{r}{m}\right)^{mt}$

If m tends to infinity,

$$A = \lim_{m\to\infty} P\left[\left(1+\frac{r}{m}\right)^{m}\right]^{t}$$

$$A = \lim_{m\to\infty} P\left[\left(1+\frac{r}{m}\right)^{\frac{m}{r}}\right]^{rt}$$

Let x=r/m, then as $m \to \infty$ implies that $x \to 0$

$$r_e = P\left[\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right]^{rt} = Pe^{rt} \text{ as } \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \mathrm{e}$$

Continuously Compounding

CONTINUOUSLY COMPOUNDED INTEREST FORMULA

$$A = Pe^{rt}$$

- A — amount at the end
- P — Principal (amount at start)
- r — annual interest rate (as a decimal)
- t — time (in years)

ILLUSTRATIVE PROBLEMS

Mr. X wants to make an investment of Rs.5,00,000 for six years. He has two alternatives. First alternative provide him a return of 8% Compounded annually and second alternative provides him a return of 7.5% compounded semi-annually. Which alternative should he select?

An amount of Rs.2000 is invested at an annual rate of 8% p.a. compounded continuously. Find out amount at the end of 5 year.

A man made a deposit of Rs.5,000 in a saving account. The deposit was left to accumulate at 6% compounded quarterly for the first 5 years and 8%compounded semi-annually for next 8 years. Find the amount at the end of 13 years.

Solution :-

Let the amount after t years be ₹ A

$A = Pe^{rt}$, as interest rate is compounding continuously.

$A = 2000\, e^{.08 \times 5}$

$A = 2000\, (e^{0.4})$ → This value we can take from e^x table

$A = 2000 \times 1.4918$

OR

$\boxed{A = ₹\,2983.6}$

we can also solve this using log.

Solution :-

Let P be the Principal, t be the years & A denotes the amount at the end of t years.

$A_t = P(1+r)^t$

Amount after 5 years

$A_5 = 5000\left(1 + \frac{.06}{4}\right)^{5 \times 4}$

$A_{13} = A_5 \left(1 + \frac{.08}{2}\right)^{8 \times 2}$ — next 8 years compounding on amount after 5 years

$A_{13} = 5000\left(1 + \frac{.06}{4}\right)^{20}\left(1 + \frac{.08}{2}\right)^{16}$

$= 5000\,(1.015)^{20}(1.04)^{16}$

$= 5000 \times 1.34685 \times 1.87298$

$\boxed{A_{13} = ₹\,12613.12}$

Solution :- In order to find the best alternative, let us compute the amount for each of alternative

P = Principal amount

m = conversion periods per year

t = no. of years , r = rate of interest

$n = mt$

$$A = P\left(1+\frac{r}{m}\right)^{mt}$$

First Alternative :-

$$A = 5,00,000\,(1+.08)^6 = 5,00,000(1.08)^6$$

$$= 500000 \times 1.586874323$$

$$\boxed{A = ₹\,7,93,437.16}$$

Second Alternative :-

$$A = 5,00,000\left(1+\frac{.075}{2}\right)^{2\times 6}$$

$$= 5,00,000\,(1.0375)^{12}$$

$$= 5,00,000 \times 1.55454331$$

$$\boxed{A = ₹\,7,77,727.16}$$

∴ First Alternative should be selected because it gives higher amount after 6 years of investment.

Asha Rani

Nominal and Effective Rate of Interest

In this topic we will firstly discuss about the nominal rate of interest then we will discuss about the effective rate of interest meaning, formula and examples to compute effective rate of interest.

The nominal rate of interest is the actual rate of interest which is stated on the any investment or loan.

When interest is compounded, more than one year, then the actual interest rate p.a. is lesser than the effective rate of interest.

The effective rate of interest is the equivalent annual rate of interest which is compounded annually. Further, the compounding must happen more than once every year.

Relationship between Nominal & Effective Rates

- Let r = Nominal Rate of Interest p.a.
- m = No. of conversion periods during a year
- P = Principal Amount
- r_e = Effective Rate p.a.
- After a year, $A = P\left(1+\frac{r}{m}\right)^m$ as well as $A = P(1+r_e)$
- $P(1+r_e) = P\left(1+\frac{r}{m}\right)^m$ which also mean $r_e = \left(1+\frac{r}{m}\right)^m - 1$

Force of Interest

- Force of Interest = Nominal rate of interest that is compounded continuously to result in an effective rate
- That is if m tends to infinity, r_e is force of interest $\lim_{m\to\infty} r_e$

$$r_e = \lim_{m\to\infty}\left[\left(1+\frac{r}{m}\right)^m - 1\right] = \lim_{m\to\infty}\left(1+\frac{r}{m}\right)^m - 1$$

$$r_e = \lim_{m\to\infty}\left[\left(1+\frac{r}{m}\right)^{\frac{m}{r}}\right]^r - 1 = \left[\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{\frac{m}{r}}\right]^r - 1$$

Let x=r/m, then as $m \to \infty$ implies that $x \to 0$

$$r_e = \left[\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right]^r - 1 = e^r - 1 \text{ as } \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \mathrm{e}$$

$$r_e = e^r - 1$$

Illustrative Problems

- Find the effective rate equivalent to the nominal rate 6% converted (i) monthly, (ii) continuously.

- Mr X took a loan of ₹ 2000 for 6 months. Lender deducts ₹ 200 as interest while lending. Find the effective rate of interest charged by lender.

Solⁿ :-

(i) $r_e = \left(1+\frac{r}{m}\right)^m - 1$

$r = 0.06 \quad m = 12$

$r_e = \left(1+\frac{.06}{12}\right)^{12} - 1$

$r_e = 1.0616 - 1 = 0.0616$

$r_e = 0.0616$ OR 6.16%

(ii)

$r_e = e^r - 1$

$= e^{.06} - 1$

taking the value of Table V from the back side of the Book

$r_e = 1.0618 - 1$

$r_e = 0.0618$ OR 6.18%

Solⁿ :- Interest deducted by money lender is ₹ 200

₹ 200 is the interest on ₹ 1800.

∴ rate is $\frac{200}{1800} = \frac{1}{9}$ $\quad m = 2$ (semi annual

$r_e = (1+i)^m - 1 \Rightarrow \left(1+\frac{1}{9}\right)^2 - 1$

$= 0.23456$ OR

$= 23.45\%$

Present Value of Amount

The Present Value is an amount that need to be invested at present so that a specified is received after a specified time with a given rate of interest.

- The compound amount formula $A = P(1+i)^n$
- Divided both sides by $(1+i)^n$
- $P = A(1+i)^{-n}$

Present Value of Amount at continuous compounding

The compound amount formula $A = Pe^{rt}$

- Divided both sides by e^{rt}

$$P = Ae^{-rt}$$

Illustrative Problems

- Mr. X left ₹50000 to be divided between his two daughters A and B. A's share was to amount to a certain sum of money at the end of 5 years and B's share was to amount to an equal amount at the end of 7 years. If the rate of interest is 6 % compounded annually, find the amount.

Let A's share amount to ₹A at the end of 5 years.
Then B's share also amounts to ₹A at the end of 7 years.

rate of interest is 6% compounded annually.

$\therefore$ P.V of A's amount $= A(1+.06)^{-5}$

P.V of B's amount $= A(1+.06)^{-7}$

$$A(1+.06)^{-5} + A(1+.06)^{-7} = 50,000$$

$$A\left[\underbrace{(1.06)^{-5}+(1.06)^{-7}}\right] = 50,000$$

taking the value from the P.V of ₹1 table

$$A[0.7472+0.6650] = 50,000$$

$$A[1.4122] = 50000$$

$$A = \frac{50000}{1.4122} = 35,405.75$$

Equation of Value

An equation of value is the equation which state that the sum of value on a given date for one set of obligation is same as, the sum of the value on the same date, of another set of obligation

We can also describe equation of value as...

Amount of loan taken → Amount of loan paid

At a particular time period

Focal Date

For comparing the value of two sets of obligation we need to select a date, that date is considered as focal date.

We can select any date as a focal date for comparison purpose

Illustrative Problems

A debt of ₹5000 due 5 years from now and ₹5000 due 10 years from now is to be repaid by a payment of ₹2000 in 2 years, a payment of ₹4000 in 4 years and final payment at the end of 6 years. If the interest rate in 7% compounded annually, how much is final payment?

A loan of ₹30,000 due 6 years from now is instead to be paid off by three payments: ₹5000 from now, ₹15,000 in three years and a final payment of Rs. ₹4750 at the end of n years. If the rate of interest is 6% compounded annually, find the value of n.

let ₹ x be the final payment.
We can select any date as focal date for comparison purpose.
Let the focal date is 6 years from now.
Value of old obligation at 6 yrs. from now is
$5000(1.07)^1 + 5000(1.07)^{-4}$ — Debt
Value of new obligation at 6 yrs from now is
$2000(1.07)^4 + 4000(1.07)^2 + x$ - payment
eqⁿ of value is

$$2000(1.07)^4 + 4000(1.07)^2 + x = 5000(1.07) + 5000(1.0$$

Checking these value from F.V & P.V Tables of ₹ 1

$$2000(1.3107) + 4000(1.1449) + x + 5000(1.07) + 5000($$

$$2621.4 + 4579.6 + x = 5350 + 3814$$

$$7201 + x = 9164$$

$$\boxed{x = ₹\ 1963}$$

Sol:

let the P.V. of the loan (due) must be equal to the P.V of the three payments

$$30,000 (1+.06)^{-6} = 5000 + 15000(1+.06)^{-3} + 4750 \times (1+.06)^{-n}$$

$$30,000(.70496) = 5000 + 15000(.839619) + 4750(1.06)^{-n}$$

$$21148.8 = 5000 + 12594.285 + 4750(1.06)^{-n}$$

$$4750(1.06)^{-n} = 3554.515$$

$$(1.06)^{-n} = \frac{3554.515}{4750} = .7483189$$

$$-n \log(1.06) = \log(0.7483189)$$

$$-0.025305865n = -.1259$$

$$n = 4.9751$$

∴ the final payment should be made at the end of 5 years (approx.) from now.

Asha Rani

Meaning of Depreciation

Depreciation is an accounting procedure for allocating the cost of the capital or non-current assets such as buildings , vehicles , machinery tools over their useful life. It is important to note the depreciation amount are estimate .

Depreciation expenses will allow firms to recapture the original amount of assets indeed to recover the original investment.

Depreciation can also be viewed as a decline in the value of assets due to age , wear and tear , or decreasing efficiency. All the assets , depreciate in value as they the get older.

Depreciation Related terms:

Original Cost/cost:

- The original cost of an asset is the **amount** of money paid for an asset which includes sales tax , delivery charges , installation charges

Useful life of Assest:

- Useful life is the life expectancy of the assets or the number of years the asset is expected to be used

Salvage Value:

- Salvage Value which is also Known as scrap value or trade-in value of the asset at the end of its useful life

Methods of Depreciation

- **Straight Line Method (SLM)**
- **Written Down Method (WDV)**

Straight Line Method (SLM)

This straight line method is very simple and the most common method. The amount of Depreciation is spread evenly to each year throughout the useful life of the asset.

The formulae for finding the annual Depreciation, annual rate of Depreciation and Book value are given as follows:

- Total Depreciation = C- S
- Annual Amount of Dep. = $(C - S)/n$
- Annual Rate of Depreciation =Annual depreciation/Total Depreciation
- Book Value at the end of k^{th} year = C – K(Annual Depreciation)

Written Down Method (WDV)

WDV is an accelerated method of depreciation in which higher depreciation charge is deducted in the early life of the assets and becomes smaller in the later years.

The formulae associated with this method for finding the annual rate of Depreciation and Book value and Depreciation at k^{th} period are given as follows:

- Book Value at the end of k^{th} year = $c(1 - r)^k$
- Annual Rate of Depreciation, r = $1 - \sqrt[n]{\frac{S}{C}}$
- Depreciation at k^{th} period = $r * BV_{k-1}$

Illustrative Problems

1. A computer whose cost is 10,00,000 will depreciate to a scrap value of 1,00,000 in 5 years. What is the book value of computer at the end of 4th year?

- If the reducing balance method of depreciation is used.
- If the straight-line method of depreciation is used.

2. An asset costing ₹4500 will depreciate to a scrap value of ₹500 in 10 years. Find the rate of depreciate.

Sol^n. Let the depreciation rate be r.

Then

$C(1-r)^5 = 1,00,000$

$10,00,000\,(1-r)^5 = 1,00,000$

$(1-r)^5 = 1/10$

taking log both the side

$5\log(1-r) = \log 1 - \log 10 \Rightarrow 0-1$

$\log(1-r) = -\frac{1}{5} = -0.2 = -0.2+1-1 = \bar{1}.8$

$(1-r) = AL(\bar{1}.8) = 0.631$

The Book value ~~after two~~ at the end of 4th year

$= 10,00,000\,(1-r)^4$

$= 10,00,000 \times (.631)^4$

$= ₹1,58,532.182$

(ii) As per SLM, the annual Depreciation

$= \frac{C-S}{\text{useful Life}} = \frac{10,00,000 - 1,00,000}{5}$

$= \frac{9,00,000}{5} = 1,80,000$

The Book value at the end of 4th year

$= 10,00,000 - 4(1,80,000)$

$= ₹2,80,000$

Sol^n:-

Let r be the rate of Depreciation.

$$C(1-r)^n = 500$$

$$4500(1-r)^{10} = 500$$

$$(1-r)^{10} = \frac{500}{4500} = \frac{1}{90}$$

taking log both the side

$$10 \log(1-r) = \log(1/9)$$

$$= \log 1 - \log 9$$

$$10 \log(1-r) = 0 - .9542$$

$$\log(1-r) = -.09542 + .1 - 1$$

$$\log(1-r) = \bar{1}.9046$$

$$(1-r) = AL(\bar{1}.9046)$$

$$(1-r) = 0.8028$$

$$r = 1 - 0.8028$$

$$= .1972 \text{ or } .1972 \times 100$$

$$= 19.72\%$$

∴ the depreciation rate is 19.72%.

Questions for Mathematics of Finance

Ques:1. (i) A certain sum of money is invested at 4% compounded annually. The interest for second year is ₹ 25. Find the interest for 3rd year.

(ii) A sum of money is put at compound interest for two years at 20% p.a. It would fetch ₹482 more, if the interest were payable half yearly than if it were payable yearly. Find the sum.

Ques:2. A sum of money is deposited in a bank which compound interest semiannually. The amount at the end of 4 years is ₹6333.85 and the amount became ₹8023.53 at the end of 8 years. Find the money deposited and the interest rate.

Ques:3. If a person deposit ₹2000 in a saving account that earns interest at the rate of 6% p.a. compounded continuously, what is the value of the account at the end of 3 years.

Ques:4. A person deposited ₹4000 in a bank at 6% compounded continuously. After 3 years, the rate of interest was increased to 7 % and after 5 years, the rate was further increased to 8 %. The money was withdrawn at the end of 10 years. Find the amount.

Ques:5. A man made a deposit of ₹2500 in a saving account. The deposit was left to accumulate at 6% compounded quarterly for the first 5 years and at 8% compounded semiannually for the next 8 years.

Ques:6. Distinguish between the nominal and effective rate of interest. Also establish the relationship between nominal and effective rate of interest when compounded n times a year and when compounded continuously.

Ques:7. Find the effective rate equivalent to the nominal rate 6% converted (i) monthly, (ii) continuously.

Ques:8. Find, for each of the following, the amount to which ₹100 will accumulate:

(i) At the rate of interest 12% p.a. compounded quarterly for 10 years.
(ii) At the force of interest 3% p.a. for 3.5 years.
(iii) At the effective rates of interest 3% p.a. for 10 years, 4% p.a. for 4 years and 5% p.a. for 2 years.
(iv) At the rate of interest corresponding to 3% p.a. effective rate of discount for 8 years.
(v) What constant force of interest would produce the same amount after 16 years as the rate in (iii) above.

Ques:9. Mr. Y has two investment options- either at 10% p.a. compounded semi-annually or 9.5% p.a. compounded continuously. Which option is preferable and why?

Ques:10. Mr. X left ₹50000 to be divided between his two daughters A and B. A’s share was to amount to a certain sum of money at the end of 5 years and B’s share was to amount to an equal amount at the end of 7 years. If the rate of interest is 6 % compounded annually, find the amount.

Ques:11. A debt of ₹5000 due 5 years from now and ₹5000 due 10 years from now is to be repaid by a payment of ₹2000 in 2 years, a payment of ₹4000 in 4 years and final payment at the end of 6 years. If the interest rate in 7% compounded annually, how much is final payment?

Ques:12. A person borrows ₹ 12,000. He pays ₹4000 at the end of 6 months and ₹5000 at the end of one year. What final payment should be made at the end of 2 years to settle the debt if the rate of interest is 12 % compounded semiannually.

Ques:13. A debt of ₹3000 which is due 6 years from now, is instead to be paid off by 3 payments ₹500 now, ₹1500 in 3 years and a final payment of ₹475 at the end of n years. If the rate of interest is 6 % p.a. effective, Find the value of n.

Ques:14. Mr. X took a loan of ₹ 50,000, payable with the interest at 10% p.a. compounded semiannually. If he pays ₹10,000 each at the end of first year and second year, find the balance payable at the end of third year if the rate of interest remains same.

Ques:15. The present value of ₹1000 due in 2 years at a certain nominal rate of discount, convertible semiannually, is ₹900. Find the rate of discount.

Ques:16. A computer whose cost is 10,00,000 will depreciate to a scrap value of 1,00,000 in 5 years. What is the book value of computer at the end of 4th year?

(i) If the reducing balance method of depreciation is used.
(ii) If the straight-line method of depreciation is used.

Ques:17. An asset costing ₹4500 will depreciate to a scrap value of ₹500 in 10 years. Find the rate of depreciate.

Ques:18. A machine costing ₹75000 is depreciated at the rate of 10% p.a. for the first 5 years and then at 12% p.a. for the next 3 year, both on diminishing balance basis. Find the book value at the end of 8^{th} year. Using this, also find average rate of depreciation.