Ex-9.2

→ The Production Function

A typical production function is stated as a function of the quantity of the inputs (factors) used in production.

i.e $P = f(L, K)$

P → Total o/t

L → Labour

K → Capital

→ An important prodn fn in economics is the Cobb-Douglas prodn fn defined by

$$Q = f(L, K) = aL^{\alpha} K^{1-\alpha}$$

$AP_L = \frac{Q}{L}$ $AP_K = \frac{Q}{K}$ → Avg productivity of input

$MP_L = \frac{\partial Q}{\partial L}$ → K (capital) Constant

$MP_K = \frac{\partial Q}{\partial K}$ → L (Labour) Constant

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→ Nature of Cobb-Douglas returns of prodn (ROS)

Application: to find returns to scale

$$Q = F(L, K) = AL^{\alpha} K^{\beta}$$

Put $L = \lambda L$

$K = \lambda K$

$\therefore$ $F(\lambda L, \lambda K)$

$= A(\lambda L)^{\alpha} (\lambda K)^{\beta}$

$= A . \lambda^{\alpha} L^{\alpha} . \lambda^{\beta} K^{\beta}$

$= \cancel{A\alpha}\ A \lambda^{\alpha+\beta} L^{\alpha} K^{\beta}$

$= \lambda^{\alpha+\beta} . Q$

if $\alpha + \beta = 1$ (Constant Returns to scale)

if $\alpha + \beta > 1$ (increasing Returns to scale)

if $\alpha + \beta < 1$ (decreasing Returns to scale)

$\boxed{Q. 13 \ i)}$ $Q = AL^{3/8} K^{3/8}$

let $L = \lambda L$

$K = \lambda K$

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$$\phi = A(\lambda L)^{3/8}(\lambda K)^{3/8}$$
$$= A\,\lambda^{3/8} L^{3/8} . \lambda^{3/8} . K^{3/8}$$
$$= A\,\lambda^{\frac{3}{8}+\frac{3}{8}} L^{3/8} . K^{3/8}$$
$$= A . \lambda^{\frac{6}{8}} L^{3/8} . K^{3/8}$$

or $$= \lambda^{3/4} . A L^{3/8} K^{3/8}$$

lessthan one

∴ The prodn fn gives decreasing ~~ROS~~ returns to scale.

x — x — x — x —

→ **Euler's Theorem**

Let us first discuss Homogeneous fn

$$Z = f(x, y)$$

If we multiply both variables by the same real number (say λ), the resulting function should be of the form:

$$f(\lambda x, \lambda y) = \lambda^n (f x, y)$$
or $$= \lambda^n . Z$$

eg; $f(x,y) = \dfrac{x^3+y^3}{xy}$

let $x = \lambda x$

$y = \lambda y$

$$F(\lambda x, \lambda y) = \frac{(\lambda x)^3 + (\lambda y)^3}{(\lambda x)(\lambda y)} = \frac{\lambda^3 x^3 + \lambda^3 y^3}{\lambda^2 xy}$$

$$= \frac{\lambda^3}{\lambda^2} \frac{(x^3+y^3)}{xy} = \lambda^{3-2} f(x,y)$$

$$\text{or} = \lambda^1 f(x,y)$$

Degree of homogeneity is 1

* Homogeneous fns of degree one are also called linear homogeneous fns.

→ Euler's Theorem

if $Z = f(x,y)$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

i.e. ~~Sum~~ Sum of, x times partial derivative of z w.r.t x & y times partial derivative of z w.r.t y, will be equal to n times the fn. z.

$n \rightarrow$ degree of homogeneity

$\boxed{Q.13\ ii)}$ Hint: $L\frac{\partial Q}{\partial L} + K\cdot\frac{\partial Q}{\partial K} < Q$.

$\boxed{\text{Notes:-}}$

* If question asks for diminishing / incr / Constant returns to scale $\Rightarrow$ Find n

* If question asks for incr / diminishing / Constant returns to inputs

$\rightarrow$ Find $\frac{\partial MP_L}{\partial L}$ & $\frac{\partial MP_K}{\partial K}$ & comment

* To find behavior of Marginal product of each factor

$\rightarrow$ Find $\frac{\partial MP_L}{\partial L}$ & $\frac{\partial MP_K}{\partial K}$ & comment.

* Each factor is paid a price equal to its marginal product $\Rightarrow$

$$L\frac{\partial Q}{\partial L} + K\frac{\partial Q}{\partial K}$$

EX-9.3

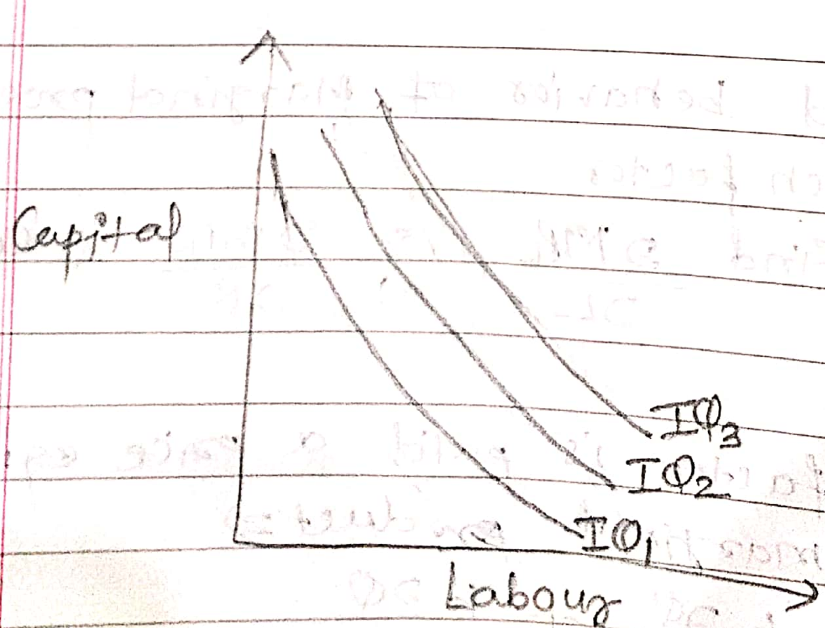
→ Isoquants (Constant product Curves)

An isoquant is a curve that shows all the combinations of inputs that yield the same level of o/t. It is a producers counterpart of the Consumers indifference curve.

Iso - quant
⇓ ⇓
Equal quantity.

i.e An isoquant represents a constant quantity of o/t.

→ Properties & Application of isoquants.



i) Isoquants are <u>downward sloping curves.</u>

$\Rightarrow$ Slope of Isoquant $= \frac{dK}{dL} < 0$

Where $\boxed{\frac{dK}{dL} = \frac{-MP_L}{MP_K}}$

ii) Isoquants are Convex to origin

$\Rightarrow$ $\boxed{\text{Convexity} = \frac{d^2K}{dL^2} > 0}$

i.e second order derivative of $\frac{dK}{dL}$ w.r.t 'L'

<u>Que</u> $Q = AL^{\alpha}K^{\beta}$, A & α, β are +ve constants
Show that the above isoquants fn is downward sloping & convex from below.

Solun:- Slope of isoquant $= \frac{dK}{dL} = -\frac{MP_L}{MP_K}$

$$= -\frac{\frac{\partial (AL^{\alpha}K^{\beta})}{\partial L}}{\frac{\partial (AL^{\alpha}K^{\beta})}{\partial K}} = -\frac{A(\alpha)L^{\alpha-1}K^{\beta}}{AL^{\alpha}(\beta)K^{\beta-1}}$$

$$= \frac{\alpha L^{(\alpha-1-\alpha)} K^{(\beta-\beta+1)}}{\beta} = \frac{-\alpha}{\beta}\left(\frac{K}{L}\right) < 0$$

Thus the given prodn fn has -ve slope.

ii) Curvature:

$$\frac{dk}{dL} = \frac{-\alpha}{\beta}\left(\frac{k}{L}\right)$$

$$\frac{d^2k}{dL^2} = \frac{-\alpha}{\beta}\left[\frac{L\frac{dk}{dL} - k(1)}{L^2}\right]$$

$$= \frac{-\alpha}{\beta}\left[\frac{L\frac{dk}{dL} - k}{L^2}\right]$$

$$\because \frac{dk}{dL} = \frac{-\alpha}{\beta}\left(\frac{k}{l}\right)$$

$$= \frac{-\alpha}{\beta}\left[\frac{\cancel{L}\left(\frac{-\alpha}{\beta}\frac{k}{\cancel{l}}\right) - k}{L^2}\right]$$

$$= \frac{-\alpha}{\beta}\left[\frac{-\alpha k}{\beta \cdot L^2} - \frac{k}{L^2}\right]$$

$$\text{or} = \frac{\alpha}{\beta}\left[\frac{\alpha}{\beta}\frac{k}{L^2} + \frac{k}{L^2}\right]$$

$$= \frac{\alpha k}{\beta L^2}\left[\frac{\alpha}{\beta} + 1\right] > 0$$

(MRTS)

→ Marginal Rate of Technical Substitution

MRTS is the number of units of K (capital) that can be sacrificed for an additional unit of L (Labour), so as to remain on same level of o/t.

It is equal to the negative slope of the isoquant.

$$\boxed{r = MRTS_{L,K} = \frac{-dK}{dL} = \frac{MP_L}{MP_K}}$$

→ Concept of total differential

$$\boxed{dQ = \frac{\partial Q}{\partial L} \cdot dL + \frac{\partial Q}{\partial K} dK}$$

Change in o/t — Change in Labour — Change in capital

- This concept will be used when there is a change in both the variables L & K and you are supposed to find out change in o/t (output).

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→ Elasticity of substitution (σ_{LK})

- It is the percentage change in the Capital-Labour ratio for every one percentage change in the $MRTS_{L,K}$ along an isoquant.

- It measures the curvature of an isoquant & thus, the substitutability between inputs.

i.e how easy it is to substitute one input for the other.

$$\boxed{\sigma_{LK} = \frac{MP_L \cdot MP_K}{Q \cdot MP_{LK}}}$$

Q = output

$$MP_{LK} = \frac{\partial (MP_K)}{\partial L} \text{ or } \frac{\partial (MP_L)}{\partial K}$$

Q. 8

$$x = Q = \left[\alpha K^{-\theta} + (1-\alpha) l^{-\theta}\right]^{-1/\theta}$$

$$\sigma_{LK} = \frac{MP_L \cdot MP_K}{Q \quad MP_{LK}}$$

$$MP_L = \frac{\cancel{-}1}{\cancel{\theta}}\left[\alpha K^{-\theta} + (1-\alpha) l^{-\theta}\right]^{\frac{-1-\theta}{\theta}}\left[\cancel{-}\cancel{\theta}(1-\alpha) l^{-\theta-1}\right]$$

$$= \left[\alpha K^{-\theta} + (1-\alpha) l^{-\theta}\right]^{\frac{-1-\theta}{\theta}}\left[(1-\alpha) l^{-\theta-1}\right]$$

$$MP_K = \frac{\cancel{-}1}{\cancel{\theta}}\left[\alpha K^{-\theta} + (1-\alpha) l^{-\theta}\right]^{\frac{-1-\theta}{\theta}}\left[\cancel{-}\cancel{\theta}\alpha K^{-\theta-1}\right]$$

$$= \left[\alpha K^{-\theta} + (1-\alpha) l^{-\theta}\right]^{\frac{-1-\theta}{\theta}}\left[\alpha K^{-\theta-1}\right]$$

$$MP_{KL} = \frac{-1-\theta}{\theta}\left[\alpha K^{-\theta} + (1-\alpha) l^{-\theta}\right]^{\frac{-1-2\theta}{\theta}}(\alpha K^{-\theta-1})\left[-\theta(1-\alpha) l^{-\theta-1}\right]$$

$$= (\alpha K^{-\theta-1})(\theta+1)\left[\alpha K^{-\theta} + (1-\alpha) l^{-\theta}\right]^{\frac{1-2\theta}{\theta}}\left[(1-\alpha) l^{-\theta-1}\right]$$

$$\sigma_{LK} = \frac{\cancel{\left[\alpha K^{-\theta} + (1-\alpha) l^{-\theta}\right]^{\frac{-1-\theta}{\theta}}}\cancel{\left[(1-\alpha) l^{-\theta-1}\right]}\cancel{\left[\alpha K^{-\theta} + (1-\alpha) l^{-\theta}\right]^{\frac{-1-\theta}{\theta}}}\cancel{\left[\alpha K^{-\theta-1}\right]}}{\cancel{\left[\alpha K^{-\theta} + (1-\alpha) l^{-\theta}\right]^{-1/\theta}}\cancel{(\alpha K^{-\theta-1})}(\theta+1)\cancel{\left[\alpha K^{-\theta} + (1-\alpha) l^{-\theta}\right]^{\frac{-1-2\theta}{\theta}}}\cancel{\left[(1-\alpha) l^{-\theta-1}\right]}}$$

$$= \frac{1}{\theta+1}$$

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EX-9.4

→ Constrained optimization - Lagrange Multiplier (Hessian Determinant)

- This concept is applied in the maximization/minimization of a fn subject to some constraints.

eg: Maximization of profit subject to level of production.

Steps:

i) Formulate Composite fn including both the given fn using Lagrange multiplier λ.

$$F(x, y, \lambda) = f(x,y) - \lambda g(x, y)$$

Where, $f(x, y)$ is a fn of two variables x & y subject to a single constraint $g(x, y) = 0$

ii) Find first order partial derivative

F_x	Put
F_y	$F_x = 0$
F_λ	$F_y = 0$
	$F_\lambda = 0$

find

iii) Second-order derivative by using:

Hessian determinant

$$\Delta = \begin{vmatrix} 0 & g_x & g_y \\ g_x & F_{xx} & F_{yx} \\ g_y & F_{xy} & F_{yy} \end{vmatrix}$$

$$= a_{11}M_{11} - a_{12}M_{12} - a_{13}M_{13}$$

minor

if $\Delta = +ve$ utility fn will be max^m

if $\Delta = -ve$ utility fn will be min^m

Ex-9.4

Q.6 $U = f(x_1, x_2) = x_1 x_2$

$g(x_1, x_2) = 4x_1 + 10x_2 - 200$

Sol^n $F(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda g(x_1, x_2)$

$= x_1 x_2 - \lambda(4x_1 + 10x_2 - 200)$

First order P.D

- $F_{x_1} = x_2 - \lambda(4) = x_2 - 4\lambda = 0$

or $\boxed{\lambda = \frac{x_2}{4}}$ — (1)

- $F_{x_2} = x_1 - 10\lambda = 0$

$\boxed{\lambda = \frac{x_1}{10}}$ — (2)

from (1) & (2)

$\frac{x_2}{4} = \frac{x_1}{10}$ or $x_2 = \frac{2x_1}{5}$

$F_\lambda = -(4x_1 + 10x_2 - 200)$

$= -4x_1 - 10x_2 + 200$

put $x_2 = \frac{2x_1}{5}$

$= -4x_1 - 10 \times \frac{2x_1}{5} + 200$

$= -4x_1 - 4x_1 + 200$

$= -8x_1 + 200$

$x_1 = \frac{200}{8} = 25 \Rightarrow \boxed{x_1 = 25}$

$x_2 = \frac{2 \times 25}{5} = \boxed{10 = x_2}$

$g(x_1, x_2) = 4x_1 + 10x_2 - 200$

$g_x = g(x_1) = 4$ i.e (P.D w.r.t x_1)

$g_y = g_{x_2} = 10$

$F(x_1, x_2) = x_1 x_2$

$F_{x_1} = x_2$, $F_{x_1 x_1} = 0$

$F_{x_2} = x_1$, $F_{x_2 x_2} = 0$

$f_{x_1 x_2} = 1$

$$\Delta = \begin{vmatrix} 0 & 4 & 10 \\ 4 & 0 & 1 \\ 10 & 1 & 0 \end{vmatrix}$$

$$= 0 - 4(-10) + 10(4-0)$$

$$= 40 + 40 = 80 > +ve.$$

Therefore utility is maxm when

$$\boxed{\begin{aligned} x_1 &= 25 \\ x_2 &= 10 \end{aligned}}$$

&

$\boxed{\text{Ex-9.5}}$

→ Applied optimization problems

Suppose a fn $z = F(x, y)$ is given

i) Find first order partial derivatives (P.D)

$$\frac{\partial z}{\partial x} = 0 \quad \text{— (1)}$$

$$\frac{\partial z}{\partial y} = 0 \quad \text{— (2)}$$

Solve (1) & (2) for x & y

ii) Find second-order P.D

$$\frac{\partial^2 z}{\partial x^2} = A$$

$$\frac{\partial^2 z}{\partial x \partial y} = B$$

$$\frac{\partial^2 z}{\partial y^2} = C$$

if $\boxed{AC - B^2 > 0}$ & $\boxed{A = +ve}$

the fn z will be maxm at these values of x & y

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& if $AC - B^2 < 0$ or (-ve) & $A = +ve$

we can ~~conclude~~ conclude that z is maxm.

* $A = +ve$ condition must always be satisfied.

* if $A = -ve$ we cannot draw any conclusion.

Ex-9.5

Q.1 $C = 2x^3 - 6xy + y^2 + 500$

$\frac{\partial C}{\partial x} = 6x^2 - 6y = 0 \rightarrow x = \sqrt{y}$

$\frac{\partial C}{\partial y} = -6x + 2y = 0$ ← substitute

$-6\sqrt{y} + 2y = 0$

$\cancel{+}6\sqrt{y} = \cancel{+}2y$

$\sqrt{y} = \frac{\cancel{2}y}{\cancel{6}} = \frac{1}{3}y$

$\sqrt{y} = \frac{1}{3}y$

or $\cancel{y} = \frac{y^{\cancel{2}}}{9}$

$\boxed{y = 9}$

$\boxed{x = \sqrt{y} = \sqrt{9} = 3}$ $\boxed{x = 3}$

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for Second-order Condition

$$\frac{\partial^2 C}{\partial x^2} = 12x = 12(3) = 36 = A$$

$$\frac{\partial^2 C}{\partial x \partial y} = -6 = B$$

$$\frac{\partial^2 C}{\partial y^2} = 2 = C$$

Since $A = 36 > 0$ &

$$AC - B^2 = 36 \times 2 - (-6)^2$$
$$= 36 > 0 \text{ or } +ve$$

Hence Cost fn is minm when

$$\boxed{x = 3}$$

$$\boxed{y = 9}$$