APPLICATION OF PARTIAL DERIVATIVES

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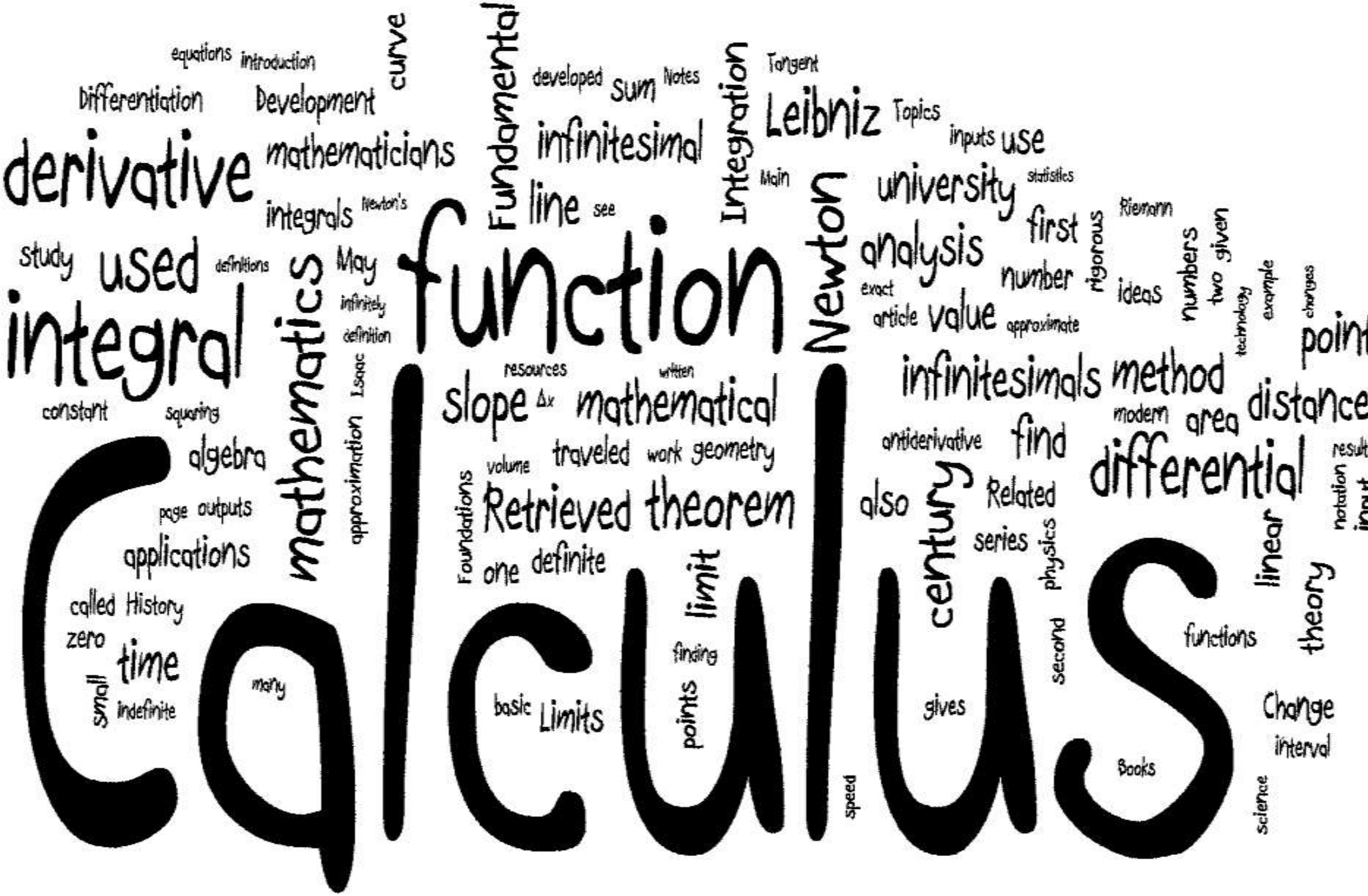
Business Mathematics
Section H, IV Semester



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To revise what we have done so far in class in Calculus…

- **Limit, function and Continuity**
- **First order derivatives – Increasing & decreasing Function and its Application**
- **Second order derivatives- Concavity, Convexity and point of inflection and its application**
- **Maxima, minima and its Application**
- **Partial derivatives- Basics**
- **Application of Partial derivates – Elasticities, Marginal Productivity.**

Session Details

This session covers : -

- Marginal Rate of Technical Substitution
- Elasticity of Substitution
- Multi-variate Optimisation
- Constrained Optimisation

Solved illustrations and Practice Handouts are provided separately.

Marginal Rate of Technical Substitution

Marginal Rate of Technical Substitution is the amount by which the quantity of one input has to be reduced when one extra unit of another input is used, so that output remains constant.

It shows the relation between inputs, and the trade-offs amongst them, without changing the level of total output.

MRTS is equal to the slope of Isoquants.

When using common inputs such as capital (K) and labour (L), the MRTS can be obtained using the following formula:

$$MRTS_L^K = \frac{dL}{dK} = \frac{MP_L}{MP_K}$$

Marginal Rate of Technical Substitution

- Let Production function is $P = f(L, K)$
- By definition $MRTS = -\frac{dK}{dL}$
- Total differential of the production function

$$dP = f_L dL + f_K dK$$

f_L and f_K are Marginal productivity of Labour & Capital which are partial derivatives of P w.r.t. L and K respectively.

- Along the an Isoquant, $dP = 0$

$$0 = f_L dL + f_K dK$$

$$-\frac{dK}{dL} = \frac{f_L}{f_K} \text{ i.e. } MRTS = \frac{f_L}{f_K}$$

Elasticity of Substitution

- Elasticity of substitution is defined as the ratio of the percentage change in the K/L ratio to the percentage change of the $MRTS_{LK}$

- $\sigma = \frac{d(K/L)/(K/L)}{d(MRTS)/MRTS} = MRTS * \frac{L}{K} * \left(\frac{1}{\frac{d(MRTS)}{d\left(\frac{K}{L}\right)}}\right)$

Illustrative Problem

Ques:1. Find the marginal rate of technical substitution and the elasticity of substitution for the following production function:

$$x = f(l, k) = \left[\alpha k^{-\theta} + (1-\alpha) l^{-\theta}\right]^{-\frac{1}{\theta}}$$

Where x is the total output obtained by using l and k units of labour and capital respectively.

Ques:- Find the MRTS and elasticity of substitution for the following production function:

$$x = f(l, K) = [\alpha K^{-\theta} + (1-\alpha) l^{-\theta}]^{-1/\theta}$$

Soln:

$$MP_L = \frac{\partial x}{\partial L} = -\frac{1}{\theta}\left[\alpha K^{-\theta} + (1-\alpha) l^{-\theta}\right]^{-\frac{1}{\theta}-1} \times (1-\alpha)(-\theta) l^{-\theta-1}$$

$$= (1-\alpha) l^{-\theta-1} \left[\alpha K^{-\theta} + (1-\alpha) l^{-\theta}\right]^{\frac{-1-\theta}{\theta}}$$

Similar way

$$MP_K = \frac{\partial x}{\partial K} = -\frac{1}{\theta}\left[\alpha K^{-\theta} + (1-\alpha) l^{-\theta}\right]^{-\frac{1}{\theta}-1} \times (\alpha)(-\theta) K^{-\theta-1}$$

$$= \alpha K^{-\theta-1} \left[\alpha K^{-\theta} + (1-\alpha) l^{-\theta}\right]^{\frac{-1-\theta}{\theta}}$$

$$MRTS = \frac{MP_L}{MP_K}$$

$$= \frac{(1-\alpha) l^{-\theta-1} \left[\alpha K^{-\theta} + (1-\alpha) l^{-\theta}\right]^{\frac{-1-\theta}{\theta}}}{\alpha K^{-\theta-1} \left[\alpha K^{-\theta} + (1-\alpha) l^{-\theta}\right]^{\frac{-1-\theta}{\theta}}}$$

$$= \left(\frac{1-\alpha}{\alpha}\right) \left(\frac{K}{l}\right)^{\theta+1}$$

$$\sigma = MRTS \cdot \frac{L}{K} \cdot \left[\frac{1}{\frac{d(MRTS)}{d(K/L)}}\right]$$

(as discussed in the previous slide)

$$\frac{d(MRTS)}{d(K/l)} = (\theta+1)\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{K}{l}\right)^{\theta}$$

$$\sigma = \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{K}{l}\right)^{\theta+1} \cdot \frac{l}{K} \cdot \frac{1}{(\theta+1)\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{K}{l}\right)^{\theta}}$$

$$= \left(\frac{K}{l}\right)^{\theta+1-\theta} \cdot \frac{l}{K} \cdot \frac{1}{\theta+1}$$

$$\sigma = \frac{1}{\theta+1}$$

Now, let's move to maxima and minima using Calculus …

More than One Independent Variables.

NOW, LET'S MOVE TO MINIMA AND MAXIMA USING CALCULUS … MORE THAN ONE INDEPENDENT VARIABLES.

MINIMA AND MAXIMA – MORE THAN ONE INDEPENDENT VARIABLE (X) WITHOUT CONSTRAINTS

✶ **Necessary Condition for Maxima and Minima:** If a function $f(x,y)$ is continuous and differentiable everywhere in the interval [a,b], and has a maxima or minima, then $f_x = f_y = 0$.

✶ **Sufficient Condition for Maxima and Minima:** If $f(x,y)$ is continuous and twice differentiable everywhere in the interval [a,b] and $f_x = f_y = 0$, then $f(x,y)$ has –

- **Maximum Value:** if $f_{xx}\, f_{yy} - f_{xy}^2 > 0$; the function will have a maximum value if $f_{xx}, f_{yy} < 0$
- **Minimum Value:** if $f_{xx}\, f_{yy} - f_{xy}^2 > 0$; the function will have a minimum value if $f_{xx}, f_{yy} > 0$

Note:

i. *Since for both minima and maxima,* $f_{xx}\, f_{yy} - f_{xy}^2 > 0$, f_{xx} and f_{yy} must be of the same sign.

ii. *If* $f_{xx}\, f_{yy} - f_{xy}^2 < 0$*, neither a maxima nor a minima exists and in such a case, there exists a* ***Saddle Point****.*

Applied Optimization Problems :-

↳ section

A firm producing two products, find the no. of units produced of each — in order to minimize T.C (joint cost)

monopolist practicing price discrimination — by selling the same product in two diff mkts at diff price — interested in finding the prices to be charged in each mkt — so as to maximize T.Profit.

Two Product Firm :- A monopolist or a firm produces two products with known ~~cost~~ mkt ...

The firm, is interested to find how many items of each be produced to have the joint profit maximization.

Illustrative Problem

Question:1. A company manufactures two types of typewriters – electrical (E) and manual (M). The revenue function of the company, in thousands, is: $R = 8E + 5M + 2EM - E^2 - 2M^2 + 20$, Determine the quantity of electrical and manual typewriters which lead to maximum revenue. Also calculate the maximum revenue.

$$R = 8E + 5M + 2EM - E^2 - 2M^2 + 20$$

$$\frac{\partial R}{\partial E} = 8 + 2M - 2E \qquad \frac{\partial R}{\partial M} = 5 + 2E - 4M$$

$$\frac{\partial R}{\partial E} = 0 \qquad \frac{\partial R}{\partial M} = 0$$

$$-2M + 2E = 8 \quad -(1) \qquad 4M - 2E = 5 \quad -(2)$$

By solving (1) & (2) equation

$$-2M + 2E = 8$$
$$4M - 2E = 5$$

$$2M = 13$$

$$M = 6.5 \quad \& \quad E = 10.5$$

$$A = \frac{\partial^2 R}{\partial E^2} = -2 \qquad C = \frac{\partial^2 R}{\partial M^2} = -4$$

$$B = \frac{\partial^2 R}{\partial E \partial M} = 2$$

$$AC - B^2 = -2 \times -4 - (2)^2 = 4 > 0$$

$A \& C < 0$

∴ The Revenue function is maximized

$$\text{Max. Revenue} = 8 \times 10.5 + 5 \times 6.5 + 2 \times 6.5 \times 10.5 - (10.5)^2 - 2(6.5)^2 + 20$$

Max. Rev = ₹ 78.25

Let us also cover one more problem given in the handouts

Ques:- A monopolist charges diff prices in two mkts where his dd function

Qty $x_1 = 21 - 0.1 p_1$, $x_2 = 50 - 0.4 p_2$ → prices

$TC = 10x + 20000$, x = total O/P. Find the price that the monopolist should charge to maximises his profit. Also verify that higher price will be charged in the Mkt having the lower price elasticity of demand.

Sol:- Let R_1 & R_2 Revenue function

$$R_1 = p_1 x_1 = p_1(21 - 0.1 p_1) = 21p_1 - 0.1 p_1^2$$

$$R_2 = p_2 x_2 = p_2(50 - 0.4 p_2) = 50 p_2 - 0.4 p_2^2$$

$TC = 10x + 2000$
$= 10(x_1 + x_2) + 2000$
$= 10x_1 + 10x_2 + 2000$
$= 10(21 - 0.1P_1) + 10(50 - 0.4P_2) + 2000$
$= 210 - P_1 + 500 - 4P_2 + 2000$
$= 2710 - P_1 - 4P_2$

$P = TR - TC$
$= (R_1 + R_2) - TC$
$= 21P_1 - 0.1P_1^2 + 50P_2 - 0.4P_2^2 - 2710 + P_1 + 4P_2$
$= 22P_1 + 54P_2 - 0.1P_1^2 - 0.4P_2^2 - 2710$

$\frac{\partial P}{\partial P_1} = 22 - 0.2P_1 = 0$ $.2P_1 = 22$
$P_1 = 110$

$\frac{\partial P}{\partial P_2} = 54 - 0.8P_2 = 0$ $.8P_2 = 54$
$P_2 = 67.5$

Also $\frac{\partial^2 P}{\partial P_1^2} = -0.2$ $\frac{\partial^2 P}{\partial P_2^2} = -0.8$

$\frac{\partial^2 P}{\partial P_1 \partial P_2} = 0$

$AC - B^2$
$-0.2 \times -0.8 - (0)^2$
$1.6 > 0$ $A = -0.2 < 0$

profit is maximum when the price charged is Rs. 110 & 67.5.

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Lagrange’s Multiplier- A Method of Constrained Optimization

Lagrange’s Multiplier…

It is a method that help us in optimizing a particular function subject to a constraint.

Let’s consider a two variable function: $f(x,y)$ which is to be optimized subject to a constraint: $g(x,y)=0$.

Also, let λ be the Lagrangian Multiplier and $\lambda g(x,y)=0$.

Then, we can define a Lagrangian Function as follows: $L(x, y, \lambda) = f(x,y) - \lambda g(x,y)$

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Lagrange's Multiplier…

Now, calculate the partial derivatives of *L(x, y, λ)* with respect to *x, y* and *λ* and put them equal to zero. And, solve for *x, y* and *λ.*

To check whether *f(x,y)* is maxima or minima, find the following Hessian Determinant-

$$H = \begin{vmatrix} 0 & g_x & g_y \\ g_x & f_{xx} & f_{xy} \\ g_x & f_{yx} & f_{yy} \end{vmatrix}$$

Lagrange's Multiplier...

If *H* is greater than zero, the *f(x,y) is maximum;*

if H is less than zero, f(x,y) is minimum.

Illustrative Problem

Question:1. Use the method of Lagrange Multiplier to find out the maximum value of

$$\mathbf{f}(x, y) = xy$$

subject to constraint:

$$g(x, y) = x + y - 5000 = 0$$

Sol^n :- We firstly construct the function as F

$$F(x,y\,\lambda) = f(x,y) - \lambda g(x,y) = xy - \lambda(x+y-5000)$$

$$\frac{\partial F}{\partial x} = y-\lambda \qquad \frac{\partial F}{\partial y} = x-\lambda \qquad \frac{\partial F}{\partial \lambda} = -x-y+100$$

The necessary condition for optimization are

$$y-\lambda = 0 \quad -(1) \qquad x-\lambda=0 \quad -(2) \qquad -x-y+5000 = 0 \quad -(3)$$

$$y=\lambda \quad , \quad x=\lambda$$

By subsituting x & y in equation -(3)

$$-\lambda-\lambda+5000 = 0$$

$$-2\lambda = -5000$$

$$\lambda = 2500$$

$$x = y = \lambda = 2500$$

The second order sufficient condition

$$\Delta = \begin{vmatrix} 0 & g_x & g_y \\ g_x & F_{xx} & F_{xy} \\ g_y & f_{yx} & f_{yy} \end{vmatrix}$$

$$\frac{\partial^2 F}{\partial x^2} = 0 \quad \frac{\partial^2 F}{\partial y^2} = 0 \quad \frac{\partial^2 F}{\partial x \partial y} = 1 \quad \frac{\partial g}{\partial x} = 1 \quad \frac{\partial g}{\partial y} = 1$$

$$\Delta = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 270$$

∴ Function is maximized

What is the Interpretation of Lagrangian Multiplier - λ?

- It represents – *Increase in the Value of Objective Function if the constraint is relaxed by one unit.*

- It is called – Marginal Contribution of 'relaxation' of the constraint by one unit.

Class Discussion Questions of Partial Derivative

Basic Questions

Ques:1. Find out the Second order Partial Derivatives of $z = 2x^2y^2 + 3xy^2 + y^3$.

Ques:2. Show that function $F(x,y) = \left[ax^{-\beta} + by^{-\beta}\right]^{-\frac{1}{\beta}}$ is homogeneous of degree 1.

Ques:3. Verify Euler's Theorem for $z = x^2e^{y/x}$.

Ques:4. Calculate the first and second order differentials of the function:

$$y = 3x^2 + xy - 2y^2$$

Ques:5. Examine the following function for local maxima and minima:

$$z = f(x,y) = 10x + 20y - 2xy - 3x^2 - 2y^2$$

Ques:6. Examine the function $z = f(x,y) = -x^3 + 27x - 4y^2$ for local extrema.

Application of Partial Derivative

Ques:7. If x_1 and p_1 are the demand and prices of tea, x_2 and p_2 are demand and price of coffee and demand function are given by:

$x_1 = p_1^{-1.7}p_2^{0.6}$ and $x_2 = p_1^{0.4}p_2^{-0.8}$

Calculate the two cross elasticities of demand and point out whether the commodities are competitive or complementary.

Ques:8. Given the demand functions of two commodities as:

$Q_1 = 2000 + \frac{400}{p_1+3} - 50p_2$ and $Q_2 = 2000 + \frac{500}{p_1+4} - 100p_2$

(i) Find the nature of commodities.
(ii) Calculate the four partial elasticities of $p_1 = 5$ and $p_2 = 1$.

Ques: 9. The joint demand functions of two products are $x_1 = 2p_1^{-0.6}p_2^{0.8}$ and $x_2 = 3p_1^{0.7}p_2^{-0.5}$, where x_1 and x_2 are the units demanded of the two products when their prices are Rs. p_1 and Rs. p_2 per unit respectively. Find four partial price elasticities at the particular prices p_1 and p_2. Also find the % changes in the quantity demanded when:

(i) p_1 increases by 2% from the existing level.

(ii) p_2 increases by 2% from the existing level.

Ques:10. The production function of a commodity is: $Q = 10L - 0.1L^2 + 15K - 0.2K^2 + 2KL$ where L is labour, K is capital and Q is production.

(i) Calculate the marginal products of the two inputs when 10 units of each labour and capital are used,
(ii) If 10 units of capital are used, what is the upper limit for the use of labour which a rational producer will never exceeds.

Ques:11. Suppose a firm has a production $x = [aL^4 + bK^4]^{1/2}$, 0<a<1, 0<b<1, where L and K are labour and capital respectively. Find the marginal product of Labour and marginal product of capital. Compute the degree of homogeneity and verify Euler's Theorem. What is the nature of returns to scale?

Ques:12. Define homogeneous function. Find out the degree of homogeneity of the following production function:

$$Q^{-\beta} = aK^{-\beta} + bL^{-\beta},$$

where Q represents the output, L and K denotes the factors of production labour and capital respectively.

Ques:13. For the following production function, show that the marginal products depend upon only on the ratio of factors

$$x = f(l,k) = \frac{2hlk - \alpha l^2 - \beta k^2}{cl + dk}$$

Also, determine the degree of homogeneity and verify Euler's theorem for the function.

Ques:14. The following is a linear homogeneous production function, where X,L, K represents output, labour and capital respectively: $X = \sqrt{aL^2 + 2hLK + bK^2}$, show that the sum of L times the marginal product of labour and K times the marginal product of capital equals to total product.

Ques:15. A production function is given by $Q=AL^{1/3} K^{1/3}$, where L and K denotes the factors of production labour and capital respectively

(i) Find the behaviour of marginal product of each factor.
(ii) What is the nature of returns to scale?

(iii) Show that the total product is not exhausted if each factor is paid a price equal to its marginal product.

Ques:16. At a certain factory, daily output is $Q = 60L^{1/3}K^{1/2}$ units, where L indicates the size of the labour force and K denotes capital investment. Use the concept of total differential to estimate the %by which the daily output change if the capital investment and labour force are both increased by 3 %. What is the nature of returns ton scale in this case?

Ques:17. If the production functions is: $X = 2ALK - BL^2 - CK^2$, find dK/dL and d^2K/d^2L. Under what condition the isoquants are downward sloping and convex to the origin?

Ques:18. Find the marginal rate of technical substitution and the elasticity of substitution for the following production function:

$x = f(l, k) = [\alpha k^{-\theta} + (1 - \alpha)l^{-\theta}]^{-1/\theta}$

Where x is the total output obtained by using l and k units of labour and capital respectively.

Ques:19. Use the method of Lagrange Multiplier to find out the maximum value of

$$f(x, y) = xy$$

subject to constraint:

$$g(x, y) = x + y - 5000 = 0$$

Ques:20. A producer desires to minimize his cost of production C = 2L + 5K, where L and K are inputs, subject to production function Q = LK. Use the method of Lagrange Multiplier to find out the optimum combination of inputs if total output is 40 units.

Ques:21. A company manufactures two types of typewriters – electrical (E) and manual (M). The revenue function of the company, in thousands, is:

$R=8E+ 5M + 2EM - E^2 - 2M^2 +20$, Determine the quantity of electrical and manual typewriters which lead to maximum revenue. Also calculate the maximum revenue.

Ques: 22. A firm sells two products A and B, their joint demand functions are

$x_1 = 175 - 4p_1 - p_2$ and $x_2 = 90 - 2p_1 - 3p_2$, where x_1 and x_2 are the units demanded for the two products when their prices are given as $p_1$and p_2 per unit respectively. Determine the prices which should be charged to maximize total revenue of the two products. Also, find the maximum revenue.

Ques: 23. A firm produces two items X_1 and X_2. The market Prices are given by $p_1 = 100 - 2x_1$ and $p_2 = 125 - 3x_2$. The cost of production is $12x_1 + 11x_2 + 4x_1x_2$ for producing x_1 and x_2 items. Find how many items of each should be produced to have the joint profit maximum?

Ques: 24. A monopolist charges different prices in the two markets where his demand function are $x_1 = 21 - 0.1p_1$ and $x_2 = 50 - 0.4p_2$, p_1 and p_2 being prices and x_1 and x_2 are quantities demanded. His total cost function is TC = 10x + 2000, where x is total output. Find the prices that the monopolist should charge to maximize his profit. Also verify that the higher price will be charged in the market having the lower price elasticity of demand.

Ques: 25. A monopolist discriminates prices b/w two markets and the average revenue functions for the two markets are:

$AR_1 = 53 - 4Q_1$ and $AR_2 = 29 - 3Q_2$

And his total cost function is C + 5Q. Find the profit maximizing outputs and prices in the two markets. What are these values when there is no price discrimination?