Riemann Integration

Integration is concerned with the problem of finding the area of a region under a curve.

Integral based on antiderivative is called Newton-Leibniz (N-L) Integral [already discussed in class]

Other kind of Integral based on exhaustion method is a Riemann Integral.
For a continuous functions both integrals will give the same result.

Method

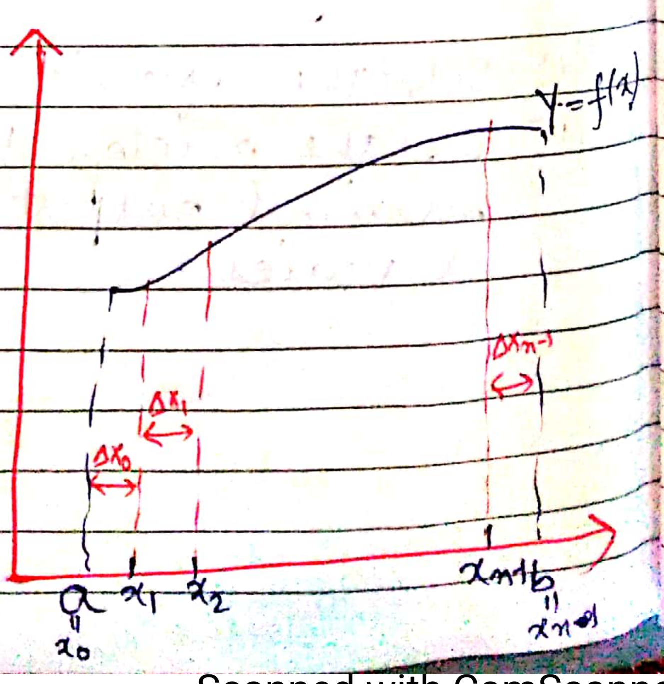
Let f be a bounded function in the integral $[a, b]$. Let n be a natural number. Subdivide $[a, b]$ into n parts by choosing pts $x_0, x_1, --, x_n$

st $a = x_0 < x_1 < x_2 ---- < x_{n-1} < x_n = b$

Width of any part is given by

$\Delta x_i = x_{i+1} - x_i$

$i = 0, 1, --, n-1$



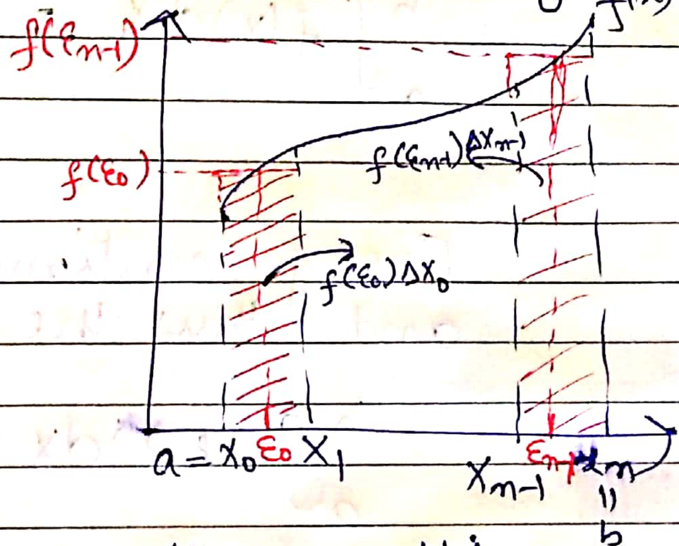
Eg → $\Delta x_0 = x_1 - x_0$
$\Delta x_1 = x_2 - x_1$
⋮
$\Delta x_{n-1} = x_n - x_{n-1}$

Choose any arbitrary number ε_i in each interval $[x_i, x_{i+1}]$.

So, ε_0 is a number chosen in the interval $[x_0, x_1]$; ε_1 is chosen in $[x_1, x_2]$, and so on -- ε_{n-1} is chosen in $[x_{n-1}, x_n]$

Now $f(\varepsilon_0)\Delta x_0$ represents the area of rectangle with width Δx_0 and height $f(\varepsilon_0)$

Similarly $f(\varepsilon_{n-1})\Delta x_{n-1}$ represents the area of rectangle with width Δx_{n-1} and height $f(\varepsilon_{n-1})$



For each ε_i and Δx_i, there will be total n rectangles which are formed.

Total sum of area of all rectangles is given by

$$f(\varepsilon_0)\Delta x_0 + f(\varepsilon_1)\Delta x_1 + \cdots + f(\varepsilon_{n-1})\Delta x_{n-1}$$

This is called Riemann sum associated with function f. This sum depends on f as well as the subdivision and on the choice of the different ε_i

$$\text{Sum} = \sum_{i=0}^{n-1} f(\varepsilon_i)\Delta x_i$$

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When n approaches infinity [means infinitely many partition of the interval [a, b]] and simultaneously the largest of the numbers $\Delta x_0, \Delta x_1, \ldots \Delta x_{n-1}$ approach 0, the limit of the sum exist. Then f is called Riemann integrable (R integrable) in the interval [a, b].

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=0}^{n-1} f(\xi_i)\,\Delta x_i$$

Every continuous function is R integrable and thus the R integral satisfies

$$\int_a^b f(x)\,dx = f(b) - f(a)$$

Question Let $f(x) = x^2 + 1$ for $0 \leq x \leq 3$. Partition the interval [0, 3] into n sub intervals of equal length (Regular partition). Find the Riemann sum and evaluate the limit as $n \to \infty$ and show limit $= \int_0^3 f(x)\,dx$.

Ans $\Delta x_0 = \Delta x_1 = \ldots = \Delta x_{n-1} = 3/n$ [Equal lengths]

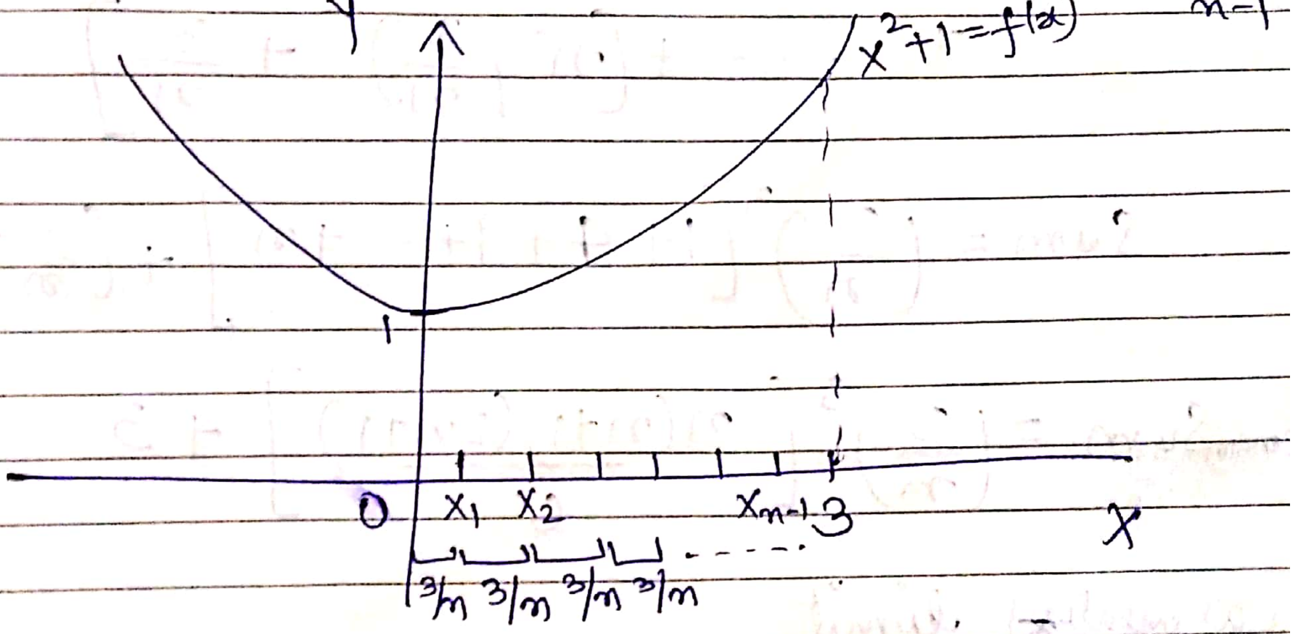
Since we can take any value of ξ_i

Let $\xi_0 = x_1$, $\xi_1 = x_2$ - - - $= \xi_n = x_n$

[As ξ_0 have to take any value in the interval $[x_0, x_1]$. Let it take the last value],

Riemann Sum $= \sum_{k=1}^{n} f(x_k) \cdot \Delta x_{k-1}$

$= f(x_1)\Delta x_0 + f(x_2)\Delta x_1 + - - - + f(x_n)\Delta x_{n-1}$



~~Bu~~ $\xi_0 = X_1 = \frac{3}{n}$

$\xi_1 = X_2 = 2(3/n)$

⋮

$\xi_{n-1} = n(3/n) = 3.$

$\text{Sum} = f\left(\frac{3}{n}\right)\Delta X_0 + f\left(2\left(\frac{3}{n}\right)\right)\Delta X_1 + - - - +$

$f\left(n\left(\frac{3}{n}\right)\right)\Delta X_{n-1}$

As $f(x) = x^2+1$ so $f\left(\frac{3}{n}\right) = \left(\frac{3}{n}\right)^2+1$

$$Sum = \frac{3}{n}\left[\left(\frac{3}{n}\right)^2+1\right] + \frac{3}{n}\left[\left(2\left(\frac{3}{n}\right)\right)^2+1\right]$$

$$+ \text{----} + \frac{3}{n}\left[\left(n\left(\frac{3}{n}\right)\right)^2+1\right]$$

$$Sum = \left[\left(\frac{3}{n}\right)^3 + \frac{3}{n}\right] + \left[4\left(\frac{3}{n}\right)^3 + \frac{3}{n}\right] + \text{---}$$

$$+ \text{---} + \left[n^2\left(\frac{3}{n}\right)^2 + \frac{3}{n}\right]$$

$$Sum = \left(\frac{3}{n}\right)^3\left[1+4+9+\text{--}+n^2\right] + \left[\frac{3}{n}+\text{---}+\frac{3}{n}\right]$$

$$\text{Rieman Sum} = \left(\frac{3}{n}\right)^3\left[\frac{n(n+1)(2n+1)}{6}\right] + 3$$

Calculating limit

$$\lim_{n\to\infty}\left(\frac{3}{n}\right)^3\left[\frac{n(n+1)(2n+1)}{6}\right] + 3 = 12.$$

which gives area under the curve from [0,3]

→ Checking the result with N-L integral as f(x) is a continuous fn

$$\int_0^3 (x^2+1)\,dx = \left[\frac{x^3}{3}+x\right]_0^3 = 12$$

Both integrals gave the same area.